YEAR 11 Mathematics Extension 1 – 2005 Yearly Examination (Term 4)

Question 1. **Marks**

(a) An arc of length 10 cm subtends an angle of θ radians at the centre of a circle of radius 5 cm. **2**
Find the value of θ correct to the nearest minute.

(b) Find the equation of the curve passing through the point $(1, 3)$ with a gradient function of $(x+1)(x-5)$. **3**

(c) Find the primitive function of

(i) $2\sin 4x$. **2**

(ii) $\dfrac{x+3}{x^2+5}$. **3**

Question 2. **[START A NEW PAGE]**

(a) The point A lies on the line $3x+2y=24$. A line, perpendicular to the x-axis, is drawn through point A and meets the x-axis at B.

(i) If B is $(a, 0)$, find the coordinates of A in terms of a. **1**

(ii) The triangle bounded by the lines AB, $3x+2y=24$ and the x-axis has an area of 27 square units, find the coordinates of A. **3**

(b) (i) Given that A is $(-3, 7)$, B is $(-2, 12)$, C is (x, y) and D is $(2, 8)$. **1**
Find the coordinates of C if $ABCD$ is a rhombus.

(ii) Hence, find the area of $ABCD$. **3**

(c) Find the integral of $4e^x+\sqrt{x}$. **2**

Question 3. **[START A NEW PAGE]**

(a) (i) Express $\sqrt{3}\sin\theta + \cos\theta$ in the form $A\sin(\theta + \alpha)$, where $A > 0$ and $0 < \theta < 2\pi$. **2**

(ii) Find the minimum value of $\sqrt{3}\sin\theta + \cos\theta$ and determine when this minimum first occurs for $\theta \geq 0$. **2**

(iii) Neatly sketch $y = \sqrt{3}\sin\theta + \cos\theta$, for $0 \leq \theta \leq 2\pi$, clearly showing showing all important feature. **3**

(b) Using the *t*-results, solve $\cos A + \sqrt{3}\sin A = -1$, for $0 \leq A \leq 2\pi$ **3**

Question 4. **[START A NEW PAGE]** **Marks**

(a) Find the exact area bounded by $y = \cos^{-1} x$, the *x*-axis and the lines at $x = 0$ and $x = \dfrac{1}{\sqrt{2}}$. **3**

(b) Prove that $\tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{4}$. **3**

(c) Find $\displaystyle\int_1^3 \frac{(x-1)}{(x+1)^3}\,dx$, using the substitution $u = x + 1$. **4**

Question 5. **[START A NEW PAGE]**

(a) If $A = \sin^{-1}\left(\frac{5}{13}\right)$, find the value of $\sin 2A$. **2**

(b) Given $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$, and $p + q = 2\sqrt{3}$. **4**
Find the angle between the chord *PQ* and the axis of the parabola.

(c) *P* is a point on $x^2 = 12y$ and *O* is the Origin. *Q* is the foot of the perpendicular from the focus, *S*, of the parabola to *OP*. **4**
Show that the locus of *Q* is given by $x^2 + y^2 - 3y = 0$.

Question 6. **[START A NEW PAGE]**

(a) Find the inverse function of $y = 3 + \ln x$. **1**

(b) Differentiate $y = \frac{1}{2}\tan^{-1} x$ with respect to *x*. **1**

(c) Neatly sketch $y = \frac{1}{2}\tan^{-1} x$ and its derivative. **3**

(d) Find the domain and range of $y = \sin^{-1}\left[\frac{1}{2(1+x^2)}\right]$. **3**

(e) Neatly sketch $y = \sin^{-1}\left[\frac{1}{2(1+x^2)}\right]$. **2**

THE END

T4 – Bunt Solutions – 2005

QUESTION 1

a) $l = r\theta$

$\theta = \frac{10}{5} = 2$ (1)

$\theta = 2 \times \frac{180}{\pi} = 114°35'$ (1)

b) $m = (x+1)(x-5)$

$= x^2 - 4x - 5$ (1)

$\therefore y = \frac{1}{3}x^3 - 2x^2 - 5x + c$ (1)

sub in $(1,3)$

$3 = \frac{1}{3} - 2 - 5 + c$

$\therefore c = -1\frac{2}{3}$ (1)

$\therefore y = \frac{1}{3}x^3 - 2x^2 - 5x - \frac{5}{3}$

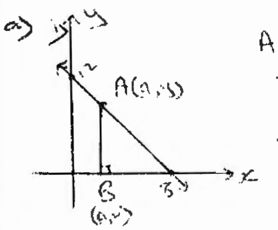
c) i) $-\frac{1}{2}\cos 4x + c$ (2)

ii) $\int\left(\frac{x}{x^2+5} + \frac{3}{x^2+5}\right)dx$

$= \frac{1}{2}\ln(x^2+5) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$ (1) (1) (1)

QUESTION 2

a)



A lies on

$3x + 2y = 24$

$\therefore 3a + 2y = 24$

$2y = 24 - 3a$

$y = 12 - \frac{3}{2}a$ (1)

$\therefore A\ (a, 12 - \frac{3}{2}a)$

ii) $A = \frac{1}{2}bh$

$27 = \frac{1}{2}(8-a)(12 - \frac{3}{2}a)$

$54 = 96 - 12a - 12a + \frac{3}{2}a^2$ (1)

$0 = \frac{3}{2}a^2 - 24a + 42$

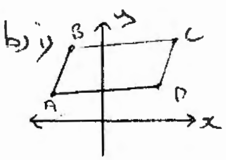
$0 = a^2 - 16a + 28$

$0 = (a-14)(a-2)$

$\therefore a = 14$ or $a = 2$ (1)

$\therefore A\ (2,9)$ or $(14,-9)$ (1)

b) i)



C $(3,13)$ (1)

ii) $A = \frac{1}{2}xy$ (1) (1)

$= \frac{1}{2}(\sqrt{16+16})(\sqrt{36+36})$

$= 24\text{ units}^2$ (1)

c) $\int(4e^{2x} + \sqrt{x})dx$

$= 2e^{2x} + \frac{2}{3}x^{3/2} + c$

QUESTION 3

a) i) $\sqrt{3}\sin\theta + \cos\theta \equiv A\sin(\theta + \alpha)$

$= A\sin\theta\cos\alpha + A\cos\theta\sin\alpha$

$\frac{\sqrt{3}}{A} = \cos\alpha$ $\quad \frac{1}{A} = \sin\alpha$

$\frac{3}{A^2} + \frac{1}{A^2} = 1$

$A^2 = 4$ (1)

$A = 2$ as $A > 0$

$\tan^{-1}\frac{1/2}{\sqrt{3}/2} = \alpha$

$\alpha = \frac{\pi}{6}$ (1)

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + \frac{\pi}{6})$

ii) Minimum value is -2 (1)

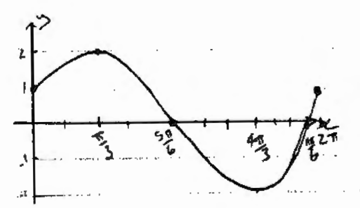
first occurs when $\sin(\theta + \frac{\pi}{6}) = -1$

$\theta + \frac{\pi}{6} = \frac{3\pi}{2}$

(1) $\theta = \frac{4\pi}{3}$

iii) $y = \sqrt{3}\sin\theta + \cos\theta$

$\therefore y = 2\sin(\theta + \frac{\pi}{6})$



(1) for endpoints

(1) intercepts

(1) range/shape

b) $\cos A + \sqrt{3}\sin A = -1$

check for $A = \pi$

$\cos\pi + \sqrt{3}\sin\pi = -1$

$-1 + 0 = -1$

true (1)

$\therefore \pi = A$ is a soln.

$1 - t^2 + 2\sqrt{3}t = -1 - t^2$

$2\sqrt{3}t = -2$

$t = \frac{-1}{\sqrt{3}}$

$\therefore \tan\frac{A}{2} = \frac{-1}{\sqrt{3}}$ (1)

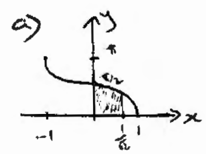
$\frac{A}{2} = \frac{5\pi}{6}$

$\therefore A = \frac{5\pi}{3}$ (1)

$\therefore$ soln. set is $A = \frac{5\pi}{3}$ or π

QUESTION 4

a)



$\text{Area} = \int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}x\,dx$ (1)

$= \int_{\pi/4}^{\pi/2} \cos y\,dy + \left(\frac{1}{\sqrt{2}} \times \frac{\pi}{4}\right)$

$= \left[\sin y\right]_{\pi/4}^{\pi/2} + \frac{\pi}{4\sqrt{2}}$ (1)

$= \sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \frac{\pi}{4\sqrt{2}}$

$= (1 - \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}})\text{ units}^2$ (1)

b) $\tan\frac{\pi}{4} = \tan(A+B)$

where $A = \tan^{-1}(\frac{1}{3})$ & $B = \tan^{-1}(\frac{1}{2})$

$\tan A = \frac{1}{3}$ $\therefore 0 < A < \frac{\pi}{4}$

$\tan B = \frac{1}{2}$ $\quad 0 < B < \frac{\pi}{4}$ (1)

$$\text{RHS} = \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$ (1)

$\therefore \tan\frac{\pi}{4} = 1$

$\tan^{-1} 1 = \frac{\pi}{4}$ (1)

$\therefore \tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{2}) = \frac{\pi}{4}$

c) $\int_1^3 \frac{x-1}{(x+1)^3}\,dx$ $\quad u = x+1$, $\frac{du}{dx} = 1$

when $x=1, u=2$
$x=3, u=4$

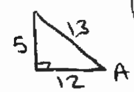
$$\therefore \int_2^4 \frac{u-2}{u^3}\,du = \int_2^4 (u^{-2} - 2u^{-3})\,du$$
$$= \left[\frac{-1}{u} + \frac{1}{2}u^{-4}\right]_2^4$$
$$= \left(\frac{-1}{4} + \frac{1}{256}\right) - \left(\frac{-1}{2} + \frac{1}{32}\right)$$
$$= \frac{-63}{256} + \frac{15}{32}$$
$$= \frac{57}{256}$$

QUESTION 5

a) $A = \sin^{-1}(\frac{5}{13})$

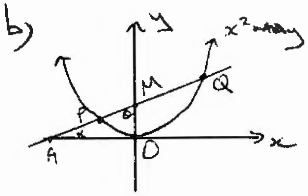
$\sin A = \frac{5}{13}$ for $-\frac{\pi}{2} \le A \le \frac{\pi}{2}$

but $\sin A > 0$, $\therefore 0 \le A \le \frac{\pi}{2}$ (1)



$$\sin 2A = 2\sin A\cos A = 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) = \frac{120}{169}$$ (1)

b)



$m = \tan\alpha$

$$\therefore \frac{a(q^2-p^2)}{2a(q-p)} = \tan\alpha$$ (1)

$$\frac{p+q}{2} = \tan\alpha$$

$$\frac{2}{2} = \tan\alpha \quad (\text{as } p+q=2)$$

$1 = \tan\alpha$

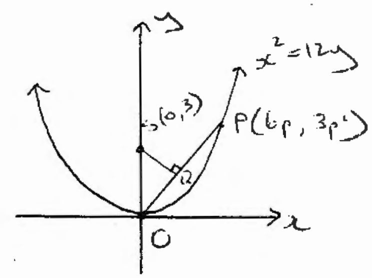
$\therefore \alpha = \frac{\pi}{4}$ (1)

(1) $\angle AOM = \frac{\pi}{2}$ (y-axis is perpendicular to x-axis)

(1) $\therefore \theta = \frac{\pi}{4}$ (angle sum of ΔAOM is π).

$\therefore PQ$ makes constant angle of $\frac{\pi}{4}$ with the axis of parabola $x^2 = 4ay$.

c)



eqn. of PO is: $y - 0 = \frac{3p^2}{6p}(x - 0)$

$y = \frac{p}{2}x$ ... (1)

eqn. of SQ is: $y - 3 = \frac{-2}{p}x$

$y = \frac{-2}{p}x + 3$ ... (2)

solve (1) & (2) to get Q

$$\frac{p}{2}x = \frac{-2}{p}x + 3$$
$$x\left(\frac{p}{2} + \frac{2}{p}\right) = 3$$
$$x\left(\frac{p^2+4}{2p}\right) = 3$$
$$\therefore x = \frac{6p}{p^2+4}$$ (1)
$$\therefore y = \frac{p}{2}\left(\frac{6p}{p^2+4}\right) = \frac{3p^2}{p^2+4}$$

$\therefore Q\left(\frac{6p}{p^2+4}, \frac{3p^2}{p^2+4}\right)$ (1)

Locus of Q is $x^2 + y^2 - 3y = 0$

$$\text{LHS} = x^2 + y^2 - 3y = \frac{36p^2}{(p^2+4)^2} + \frac{9p^4}{(p^2+4)^2} - \frac{9p^2}{p^2+4}$$
$$= \frac{36p^2 + 9p^4 - 9p^2(p^2+4)}{(p^2+4)^2}$$ (1)
$$= 0$$

$\therefore$ Locus of Q is the circle $x^2 + y^2 - 3y = 0$ (1) with centre $(0, 1\frac{1}{2})$, radius $\frac{3}{2}$.

QUESTION 6

a) $y = 3 + \ln x$

inverse is $x = 3 + \ln y$

$x - 3 = \ln y$

$\therefore y = e^{x-3}$ (1)

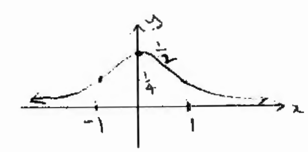
b) $y = \frac{1}{2}\tan^{-1}x$

$$\frac{dy}{dx} = \frac{1}{2(1+x^2)}$$ (1)

c) $y = \frac{1}{2}\tan^{-1}x$

($1\frac{1}{2}$)

$y = \frac{1}{2(1+x^2)}$

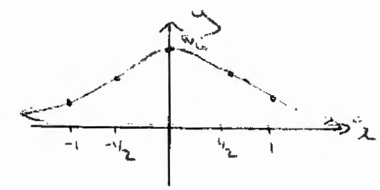


($1\frac{1}{2}$)

d) Domain: $x \in R$ (1)

Range: $0 < y \le \frac{\pi}{6}$ (2)

e)



(1) for shape ($\frac{1}{2}$) ... ($\frac{1}{2}$) ... asymptote

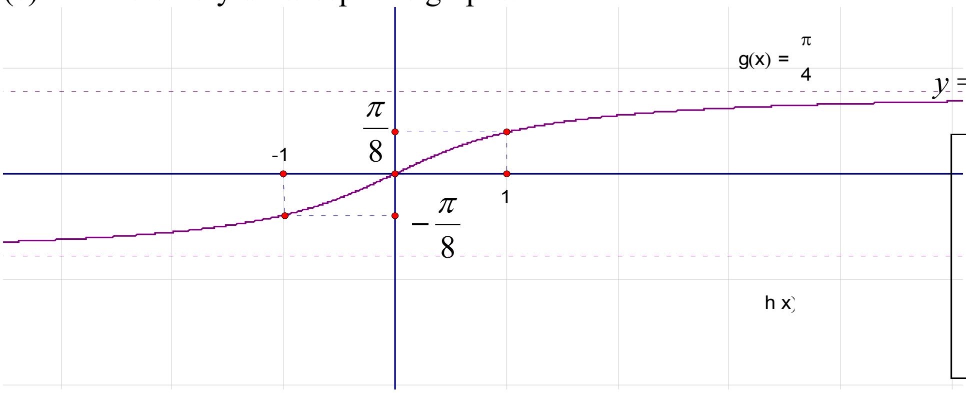
Term 4 Ext 1 Maths QUESTION 6 Marking Guidelines (L. Kim)

(a) $y = 3 + \ln x$ ➔ swap x and y

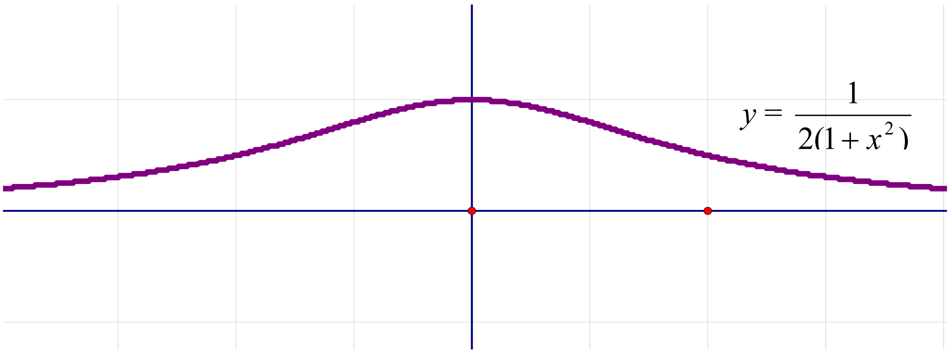
$\therefore x - 3 = \ln y$ ➔ $y = e^{x-3}$ **1 mk** No half marks awarded.

(b) $y = \frac{1}{2}\tan^{-1} x$ $\therefore \frac{dy}{dx} = \frac{1}{2(1+x^2)}$ **1 mk** No half marks

(c) Preferably draw separate graphs.



1 mk shape
½ mk asymptotes
½ mk scale
[shape wrong direction – ½ mk]



½ mk shape
½ mk y – intercept at $y = ½$.

Graph should be symmetrical.

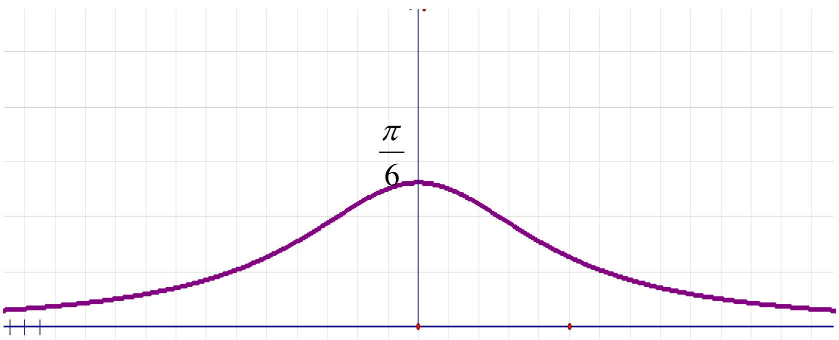
(d) From part (c) second graph, the domain of $y = \frac{1}{2(1+x^2)}$ is $\{x : x \varepsilon R\}$.

Also the range is $0 < y \leq ½$ $\therefore \sin^{-1}(0) < \sin^{-1}\left(\frac{1}{2(1+x^2)}\right) \leq \sin^{-1}(½)$

$\therefore$ FOR $y = \sin^{-1}\frac{1}{2(1+x^2)}$

DOMAIN is $\{x : x \varepsilon R\}$ 1mk and the RANGE is $0 < y \leq \frac{\pi}{6}$. 2mk- ½ mk each

(e)



½ mk shape
½ mk y – intercept at $y = \frac{\pi}{6}$
½ mk asymptote $y = 0$
½ mk scale
Graph should be symmetrical.